

GLUON CONDENSATE FROM LATTICE QCD*

Xiangdong Ji

*Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139*

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Abstract

After making some critical comments about the traditional method of extracting the gluon condensate from lattice QCD data, I present an alternative analysis. The result is more than a factor of five larger than the phenomenological value. Two closely related subjects, the effects of the infrared renormalons on the extraction and the Lepage and Mackenzie improvement on a lattice perturbation series, are also discussed.

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One of the popular approaches to study Quantum Chromodynamics (QCD) in the strong coupling region is the QCD sum rule method initiated by Shifman, Vainshtein, and Zakharov (SVZ) more than fifteen years ago [1]. The success of the approach has been demonstrated by many examples ranging from hadron masses, light-front wavefunctions to decay widths, form factors, etc. Central to the QCD sum rules is the concept of vacuum condensates which are expectation values of composite operators in the QCD vacuum. Among those, the most legend is perhaps the gluon condensate defined through the operator $F^2 = F^{\alpha\beta}F_{\alpha\beta}$, where $F^{\alpha\beta}$ is the strength of color gauge fields.

To understand the sum rule phenomenology at a more fundamental level, one has to calculate the vacuum condensates directly from the QCD lagrangian. Calculations using lattice Monte Carlo started in several groups shortly after the publication of SVZ's paper [2]. Before I comment on these calculations, it is important to point out that phenomenological condensates from fitting experimental data are in principle different from theoretical condensates that are calculated as matrix elements in the QCD vacuum. The former are extracted with the Wilson coefficients computed to a few loops, and thus may contain large uncalculated multi-loop contributions from the coefficient functions and higher-power corrections.

The traditional approach of calculating the gluon condensate (in the quenched approximation) goes like this [2]. Consider an elementary plaquette on the lattice. Calculate using Monte Carlo the trace of the plaquette as a function of the lattice spacing a , or the lattice coupling constant $\beta = 2N_c/g_0^2$. For small enough a , one has,

$$1 - \frac{1}{N_c} \text{Tr} P = \sum_{n=1} \frac{c_n}{\beta^n} + \frac{\pi^2}{12N_c} a^4(\beta) \langle \frac{\alpha_s}{\pi} F^2 \rangle + \mathcal{O}(a^6) . \quad (1)$$

The various terms on the right-hand side have different characteristic β (or a) dependences. The leading term is a series logarithmic in a (or power-like in $1/\beta$). The condensate term is quartic in a (or exponential in β). By calculating both the left-hand side and the perturbation series for a wide range of β , the condensate can be extracted by fitting the expected β dependence.

The approach shall work in principle. But it is difficult to implement in practice. Let me list a few of the practical problems with the approach:

- First, a big lattice is required to compute the plaquette average for small a . Physically the gluon condensate comes from the effects of long wavelength gluons (somewhere from 0.5 to 1 fm). The combination of a small lattice spacing and a reasonable physical size requires a big lattice. In ref. [3], it was determined that the asymptotic region, where the data behave like a sum of powers plus an exponential in β , starts from $\beta = 6.58$. Using a two-loop relation between β and a , $a = (\beta/6b_0)^{b_1/2b_0^2}/\Lambda_L \exp(-\beta/12b_0)$ with $\Lambda_L = 4.4$ MeV, $b_0 = 11/16\pi^2$ and $b_1 = 102/256\pi^2$, one finds that the coupling corresponds to a lattice spacing 0.055 fm. Thus a lattice with 8 points in spatial directions spans a physical dimension of 0.44 fm. For $\beta = 7$, the lattice size further reduces to 0.27 fm. Both lattices seem to be too small to measure non-perturbative physics.
- Second, relative to the leading term, the condensate term contributes little to Eq. (1) as a decreases. At $a = 0.055$ fm, the ratio between the leading and the condensate

terms is more than 10^3 . Thus to determine the condensate with a reasonable error at small a , one has to compute the perturbation series to large orders.

- Finally, the perturbation series in Eq. (1) actually divergences due to the infrared (IR) renormalons, that is, the coefficients of the series increase like $n!$ and with a fixed sign [4]. The presence of the renormalons complicates the extraction of the condensate in two ways. 1) The perturbation series cannot yield better accuracy beyond a certain order. 2) The difference between power and exponential behaviors in β , which has been the basis for fitting the condensate, disappears to a certain degree.

The physical significance of the gluon condensate in light of the IR renormalons has been discussed in the literature for a long time [5–7]. The upshot of these discussions is that the gluon condensate is a procedure-dependent concept. [In this sense, the status of a theoretical condensate is not much different from a phenomenological one.] When a condensate is calculated in a particular scheme designed to regularize the infrared renormalons, the coefficient functions in an operator product expansion must be calculated accordingly. A consistent method to regularize the renormalons both in the coefficient functions and condensates, generalizing the one-loop discussion in ref. [7], has recently been studied by this author [8].

In light of the above observations, I consider an alternative strategy to extract the gluon condensate from Eq. (1). Assuming the plaquette perturbation series is an asymptotic one, one would expect that the magnitude of its terms initially decreases, then reaches a minimum, and finally increases without bound. It is generally believed that the minimal term occurs at order $n \sim \beta$, with a magnitude $\sim a^4$. Thus the minimum uncertainty in the perturbation series induces an uncertainty in the gluon condensate, which can only be eliminated through a regularization of the series. However, if the uncertainty is small compared with the condensate itself, as Novikov et al. argued [5], it still makes a good deal of physical sense to extract a gluon condensate independent of the regularization scheme. To illustrate this more clearly, I schematically show in Fig. 1 the order at which the minimum term occurs as a function of β (the solid line labelled by M). I also show the order at which the size of the perturbative term is roughly that of the gluon condensate (with the solid line labelled by C). If the uncertainty on the condensate induced by IR renormalons is small, the C line would be significantly below the M line.

Now suppose the perturbation series has been calculated to some fixed order n_0 , shown by the dashed line in Fig. 1, and call its intersection with curve C , β_{\max} . Clearly, one cannot extract the gluon condensate accurately if $\beta > \beta_{\max}$. On the other hand, for a small β , the corrections from the lattice discretization and the higher-dimensional condensates in the expansion become important. Thus there exists a lower limit on β , β_{\min} , below which the extraction becomes unreliable. The existence of a window in β ($\beta_{\min} < \beta_{\max}$) depends on n_0 . So long as a window exists, one shall be able to make an approximate extraction. The error on the condensate depends on n_0 . When n_0 is greater than $n(\beta_{\min})$ on the M curve, the error is limited by the renormalon singularity.

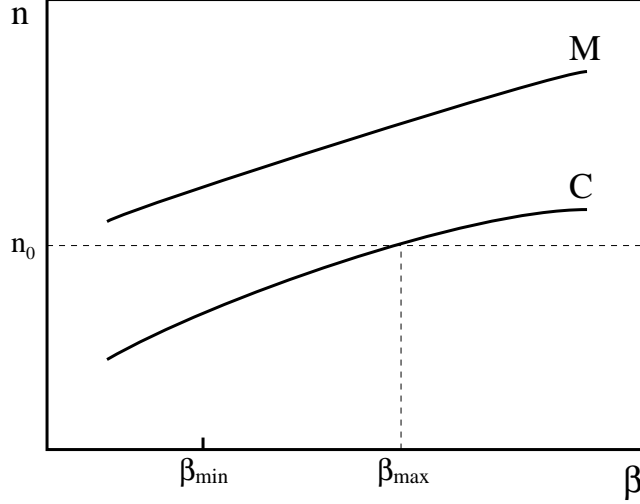


Fig. 1. See the text for the explanation of various curves.

The perturbation series for the elementary plaquette has initially been calculated to three loops by Alles, Campostrini, Feo, and Panagopoulos [9] and recently to eight loops by Di Reno, Onofri, and Marchesini using a numerical method [10]. With $n_0 = 8$, β_{\max} is somewhere around 6.5 to 7. Experiences with lattice calculations indicate a β_{\min} around 5.7. In the following analysis, I choose three lattice couplings: $\beta = 5.7, 6.0$, and 6.2 , corresponding to the two-loop lattice spacings $a = 0.15, 0.105, 0.084$ fm, respectively.

For the average of the elementary plaquette, I use the following result from the lattice Monte Carlo [3,11,12],

$$\begin{aligned} 1 - \frac{1}{N_c} \text{Tr} P &= 0.4509 \quad (\beta = 5.7) , \\ &= 0.4058 \quad (\beta = 6.0) , \\ &= 0.3861 \quad (\beta = 6.2) . \end{aligned} \quad (2)$$

Since the numerical errors are small compared with the uncertainty in perturbation series, I have ignored them. The perturbation series from Ref. [10] is,

$$\sum_n \frac{c_n}{\beta^n} = \frac{2}{\beta} + \frac{1.218}{\beta^2} + \frac{2.960}{\beta^3} + \frac{9.28}{\beta^4} + \frac{34}{\beta^5} + \frac{135}{\beta^6} + \frac{563}{\beta^7} + \frac{2488}{\beta^8} + \dots \quad (3)$$

where I have also ignored the small Monte Carlo errors. For β 's under consideration, the last term calculated is not yet the minimal term in the series. I assume for the moment the error in truncating the series is determined by the last term,

$$\begin{aligned} \sum_n \frac{c_n}{\beta^n} &= 0.4278 \pm 0.0022 \quad (\beta = 5.7) , \\ &= 0.3988 \pm 0.0015 \quad (\beta = 6.0) , \\ &= 0.3818 \pm 0.0011 \quad (\beta = 6.2) . \end{aligned} \quad (4)$$

Then the differences between the Monte Carlo data and the perturbation series are, 0.0222 ± 0.0022 , 0.0075 ± 0.0015 , 0.0045 ± 0.0011 for $\beta = 5.7, 6.0, 6.2$, respectively. In producing

the small differences, the knowledge of four, five, and six-loop terms in the series has been essential. Notice that the relative error is larger for larger β . This reflects the observation made early—to extract the condensate at larger β , one needs higher precision for the series.

The error estimate in Eq. (4) is probably optimistic. One indication is that terms in the perturbation series have the same sign and decrease slowly with increasing n . To improve the extraction, one may estimate higher-order terms in the series. In the following, I shall consider two such estimates: the Pade approximation and the leading IR renormalon approximation.

In the standard Pade approach, both $[3,4]$ and $[4,3]$ approximations produce ($[m,n]$ is the standard notation for the Pade approximation with a m -th order polynomial in the numerator and a n -th order polynomial in the denominator),

$$\begin{aligned}\sum_n \frac{c_n}{\beta^n} &= 0.4322 \quad (\beta = 5.7) , \\ &= 0.4011 \quad (\beta = 6.0) , \\ &= 0.3833 \quad (\beta = 6.2) .\end{aligned}\tag{5}$$

The differences between the Monte Carlo data and the Pade approximation are, 0.0187, 0.0047, 0.0028 for $\beta = 5.7, 6.0, 6.2$, respectively. The numbers are about two standard deviations away from the eight-loop result, indicating that the eighth-order term may not be in the asymptotic region.

Using the leading IR renormalon series constructed in Ref. [10], I find the minimal term for the plaquette series occurs somewhere around the 25th to 30th order for the present β 's. The first few terms beyond the eighth order go like this,

$$\begin{aligned}\sum_n \frac{c_n}{\beta^n} &= \dots + \frac{1.1 \times 10^4}{\beta^9} + \frac{4.8 \times 10^4}{\beta^{10}} + \frac{2.1 \times 10^5}{\beta^{11}} + \frac{9.5 \times 10^5}{\beta^{12}} + \frac{4.2 \times 10^6}{\beta^{13}} \\ &\quad + \frac{1.9 \times 10^7}{\beta^{14}} + \frac{8.9 \times 10^7}{\beta^{15}} + \frac{4.1 \times 10^8}{\beta^{16}} + \frac{1.9 \times 10^9}{\beta^{17}} + \frac{9.4 \times 10^9}{\beta^{18}} + \dots\end{aligned}\tag{6}$$

Summing the series in the renormalon approximation to the minimal term yields,

$$\begin{aligned}\sum_n \frac{c_n}{\beta^n} &= 0.4362 \quad (\beta = 5.7) , \\ &= 0.4029 \quad (\beta = 6.0) , \\ &= 0.3848 \quad (\beta = 6.2) .\end{aligned}\tag{7}$$

The differences between the Monte Carlo data and the perturbation series in the leading-renormalon approximation are, 0.0147, 0.0029, 0.0013 for $\beta = 5.7, 6.0, 6.2$, respectively. The numbers are about three standard deviation away from the sum to eight loops.

As a final result, I take the average of the two large-order estimates as the central value and take their difference as the error estimate. Thus the condensation contributions to the plaquette average are 0.0167 ± 0.0040 , 0.0038 ± 0.0018 , and 0.0021 ± 0.0015 for $\beta = 5.7, 6.0, 6.2$, respectively. These numbers can be converted to the standard form of the gluon condensate in unit of GeV^4 ,

$$\begin{aligned}
\langle \frac{\alpha_s}{\pi} F^2 \rangle &= 0.18 \pm 0.04 \quad (\beta = 5.7) , \\
&= 0.17 \pm 0.08 \quad (\beta = 6.0) , \\
&= 0.23 \pm 0.17 \quad (\beta = 6.2) .
\end{aligned} \tag{8}$$

Again, the large error for $\beta = 6.2$ indicates that the perturbation series must be calculated to higher orders in order to extract the condensate at large β .

The numbers above are consistent with that extracted in Ref. [3]. They are at least a factor of five larger than the phenomenological determination (See Ref. [13] for a discussion about the phenomenological condensate). Several factors may contribute to this discrepancy: 1) higher-orders in the plaquette series are different from what we expected, 2) quenched approximation, 3) underestimation of the lattice spacing, 4) the phenomenological condensate is contaminated by higher-order Wilson's coefficients. Clearly, it is interesting and also important to resolve the discrepancy in this high-precision confrontation between lattice Monte Carlo, QCD perturbation theory, and hadron phenomenology.

The perturbation series for the elementary plaquette (Eq. (3)) is the first that has been calculated up to more than two-loops in lattice QCD. Consequently, it is interesting to explore its convergence properties. In general, because of the large tadpole contributions, lattice perturbation series converges very slowly. The plaquette series does show such slow convergence due to an abnormal fast increase in the coefficients, as observed by Di Renzo et al. [10]. If a perturbation series has been calculated to sufficient large orders, the slow convergence is not a problem. In fact, as I will argue later that the lattice perturbation series has the virtue of producing small renormalon uncertainties.

Lepage and Mackenzie [12] have recently pointed out that the perturbative expansion with a more "physical" coupling accelerates the convergence of a lattice series. Clearly, if a series is limited to the first few terms, the acceleration improves its predictive power considerably. The plaquette series provides an excellent example for studying the effects of acceleration. To show this, let me introduce

$$F = \frac{\beta}{2} \left(1 - \frac{1}{N_c} \text{Tr} P \right) - 1 . \tag{9}$$

The series for F is,

$$\begin{aligned}
F &= 1.2755 \alpha_{s0} + 6.4920 \alpha_{s0}^2 + 42.6276 \alpha_{s0}^3 + 327.1 \alpha_{s0}^4 \\
&\quad + 2720.1 \alpha_{s0}^5 + 23758.8 \alpha_{s0}^6 + 219899 \alpha_{s0}^7 + \dots ,
\end{aligned} \tag{10}$$

where α_{s0} is the bare lattice coupling. If truncating to the leading term, the result for F , 0.1015, is only half of the full series, 0.2087, at $\beta = 6.0$. The bad approximation by the leading term is related to the large coefficient 6.4920 in the second term and, in general, the fast increase of the perturbative coefficients. However, If changing to a renormalized coupling which coincides with the one-loop $\overline{\text{MS}}$ coupling at scale $\mu = \pi/a$,

$$\alpha_{s0} = \alpha_s / (1 + 3.88 \alpha_s) , \tag{11}$$

F becomes,

$$F = 1.2755 \alpha_s + 1.543 \alpha_s^2 + 11.44 \alpha_s^3 + 49.7 \alpha_s^4 + 265.8 \alpha_s^5 + 1566 \alpha_s^6 + 9616 \alpha_s^7 + \dots . \tag{12}$$

Clearly, the convergence has been improved considerably due to the small expansion coefficients. The leading term is now 0.1584, a number much closer to the full result. This phenomenon has already been studied in Ref. [10].

Two comments can be made about the acceleration of the lattice perturbation series, both of which are related to the question of precision, important, as we have seen, in extracting the gluon condensate. First, in the expansion with a new coupling, the coefficients of the lower-order terms in general do not increase at a monotonic rate, i.e., they become less regular. Regularity is a necessary (not sufficient, of course) diagnosis for the series in the asymptotic region, where one can estimate the error of truncation using the last term included. To produce a series with regular lower-order terms, one must use some "natural" couplings with well-defined relation to the lattice coupling up to high orders. One possibility is to use couplings associated with some "physical" schemes like $\overline{\text{MS}}$. However, the relation between the lattice and $\overline{\text{MS}}$ couplings is difficult to calculate. Only recently, the relation has been computed to two-loops by Luscher and Weisz [14],

$$\alpha_{s0} = \alpha_s - 3.88 \alpha_s^2 + 7.744 \alpha_s^3 + \dots, \quad (13)$$

where α_s is again defined at scale $\mu = \pi/a$. Using a rationalized version of the above relation, $\alpha_{s0} = \alpha_s / (1 + 3.88\alpha_s + 7.310\alpha_s^2)$, I have,

$$F = 1.2755 \alpha_s + 1.543 \alpha_s^2 + 2.128 \alpha_s^3 + 29.20 \alpha_s^4 + 169.0 \alpha_s^5 + 880.5 \alpha_s^6 + 9835 \alpha_s^7 + \dots. \quad (14)$$

The coefficient of α_s^3 term is now reduced significantly, compared with that in Eq. (12). However, the irregularity has been shifted to higher-order terms. This is reflected by the fluctuation of the successive terms, which makes error-estimate difficult.

The second comment is about the intrinsic error of the series due to IR renormalons. In new expansions, the series converges faster, however, the minimal term in the series in general becomes larger. Take the above series as an example, the smallest term in the series is approximately 0.0030. Before the improvement, however, the smallest term in the series is less than 10^{-4} . This dependence of the renormalon uncertainty on expansion schemes is not well studied and is quite interesting. It may imply the existence of a scheme in which the coupling constant approaches to zero and the power series converges infinitely slowly, but, the renormalon ambiguity may be vanishingly small.

To summarize, I have critically examined the traditional method of extracting the gluon condensate from the lattice QCD data. Using the plaquette series calculated recently to eight loops, I have made a direct determination of the condensate at low β and large lattice spacings. The result is at least a factor of five larger than the phenomenological condensate. Although still higher-order terms in the perturbation series must be computed to reduce the uncertainty, the intrinsic error due to IR renormalons seems to be small. The plaquette series converges faster when expanded with more physical couplings, however, the estimation of errors becomes more difficult.

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